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Dimensional reduction and correlation functions on 3D lattice models

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Abstract. The exact calculation of an infinite number of correlation functions for general cubic and HCP anisotropic Ising models on their 'disorder variety' is described. The intra-plane correlation functions are found to be exactly the same as for a two-dimensional free-fermion Ising model. This allows us to characterise the intersection between the disorder and critical variety which displays in general anisotropic scaling with non-trivial exponents.

1. Introduction

Many exact simple solutions for lattice models of statistical mechanics called 'disorder solutions' have been obtained in the last fifteen years (Stephenson 1970, Welberry and Galbraith 1973, Verhagen 1976, Enting 1977, 1978, Rujan 1982, Peschel and Rys 1982, Dhar 1983). Most of these solutions are obtained for two-dimensional models but there are a few examples of such exact solutions for three-dimensional models (Welberry and Miller 1978, Enting 1977, Rujan 1982, Domany 1984, Jaekel and Maillard (1985).

These simple solutions are provided by some local condition bearing on the Boltzmann weight of the elementary cell generating the lattice (Jaekel and Maillard 1985). A straightforward consequence of this local condition is a certain decoupling of the spin degrees of freedom: for instance, the partition function per cell of these three-dimensional models reduces, when the model is restricted to a certain subvariety of the parameter space corresponding to this local condition, to the partition function of the isolated elementary cell generating the lattice. For correlation functions in two dimensions a similar dimensional reduction occurs: this has been illustrated on the two-dimensional example of the checkerboard Ising (or Potts) model (Dhar and Maillard 1985). In that case it has been shown that an infinite number of correlation functions can be calculated on the disorder varieties, thanks to a dimensional reduction from two to one dimensions. One purpose of this paper is to show for the example of two particular models that similar dimensional reductions occur for three-dimensional Ising models. An infinite number of correlation functions can be calculated exactly. For instance, the intra-plane correlation functions reduce to the correlation functions of a two-dimensional model (free-fermion models in the examples of the paper).

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2. Disorder condition on a cubic Ising model

The cubic three-dimensional Ising model studied here has been defined by Jaekel and Maillard (1985). The elementary cell of the model has three different coupling constants (figure 1(a)). These cells are arranged in a staggered way as indicated by figure 1(c). The Boltzmann weight of the elementary cubic cell is

$$W(K, K', L) = [K(\sigma_i\sigma_j + \sigma_j\sigma_k + \sigma_k\sigma_l - \sigma_l\sigma_i) + L(\sigma_i\tau_i + \sigma_j\tau_j + \sigma_k\tau_k + \sigma_l\tau_l) + K'(\tau_i\tau_j + \tau_j\tau_k + \tau_k\tau_l - \tau_l\tau_i)]. \tag{1}$$

We require, and this is our local disorder condition, that the Boltzmann weight W associated with the elementary cube is actually independent of the four spins σ_i at the bottom of the cube when one sums over the spin configurations of the spins τ_i at the top of the cube:

$$\sum_{\tau_i} W(K, K', L) = \lambda(K, K', L) \qquad \text{independent of } \sigma_i. \tag{2}$$

A straightforward calculation shows that this (disorder) condition is satisfied when

$$\tanh^2 L \tanh 2K + \tanh 2K' = 0. \tag{3}$$

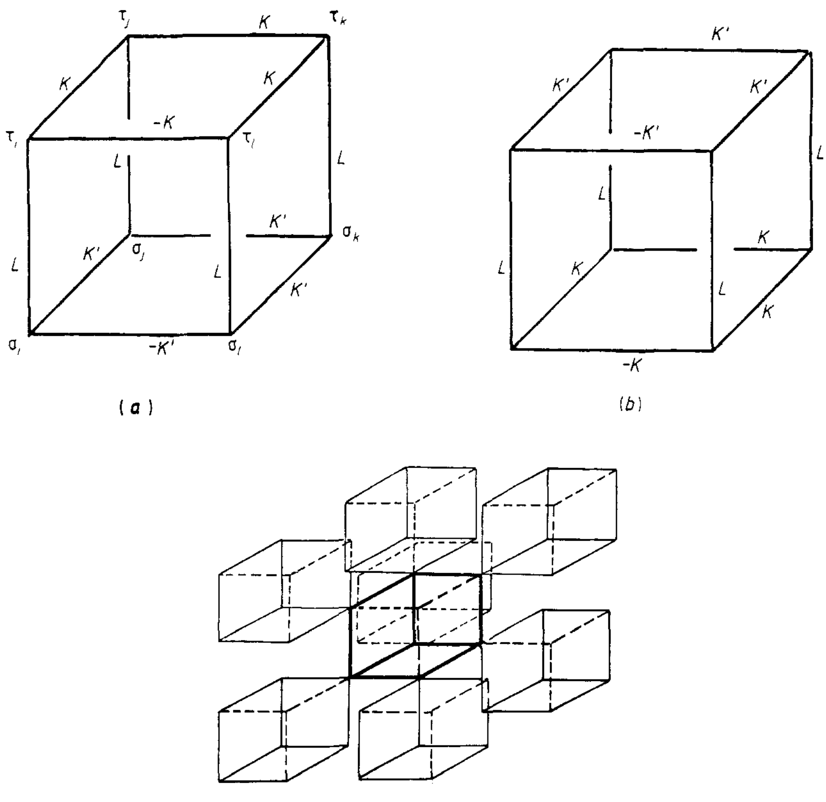


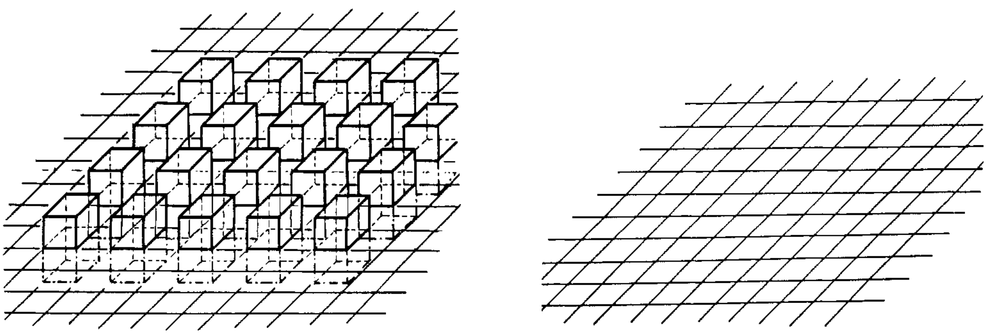
Figure 1. (a) Spins and the three coupling constants corresponding to the elementary cube of the lattice (green cube). (b) Alternative elementary cell of the lattice: the red cubes. (c) The staggering of the elementary cubic cells.

For some appropriate boundary conditions on the lattice (detailed in Jaekel and Maillard (1985)) the partial summation over all the spins at the boundary leads to the disappearance of all the elementary cubes of the first layer and one recovers the same boundary conditions for the next layer. One can iterate this procedure recursively and 'eat' the whole lattice in that way. What remains after this decimation procedure is just a multiplicative factor for each elementary cube. This leads to the following partition function per site of this cubic Ising model restricted to the disorder condition (3):

$$Z_{\text{site}(3)} = \lambda^{1/4}. \tag{4}$$

A similar decimation procedure can be performed to calculate correlation functions exactly for the three-dimensional model above when condition (3) is satisfied. This is a generalisation of the method introduced by Dhar and Maillard (1985) to calculate an infinite number of correlation functions on the checkerboard Potts (or Ising) model. For that purpose one remarks that the lattice can also be obtained from another elementary cell: the elementary cubic cell depicted in figure 1(*b*). Let us denote by 'green' and 'red' these two kinds of cubic elementary cells corresponding respectively to figure 1(*a*) and figure 1(*b*). The following obvious remark holds: when the condition (2) bearing on the red cubes is satisfied, the symmetric condition bearing on the green cubes is simultaneously satisfied. Hence when one sums over the configurations of the spins at the *bottom* of the green cube the Boltzmann weight of the green cube is independent of the configurations of the spins at the top of that cube. With appropriate boundary conditions for the bottom of the lattice (symmetric to the previous top one) one can integrate over the spin configurations from the top layers of spins downwards recursively 'eating away' the red cubes, and from the bottom layer upwards by integrating over the green cubes. Figure 2(*a*) and figure 2(*b*) show what remains of the lattice after such a decimation: an n-point intra-plane correlation function reduces to an n-point correlation function of a two-dimensional free-fermion Ising model whose elementary cell will be defined below (see figure 6).

Let us now consider n-point correlation functions that are not intra-plane correlation functions. When one encounters in such a decimation procedure a point (of some n-point correlation function) the decimation of the cube can no longer be performed. To fix ideas let us concentrate on the green cubes: equation (2) is actually satisfied



(*a*) (*b*)

Figure 2. (*a*) What remains of the cubic lattice after the last but one step of the upward and downward decimation procedure. (*b*) What remains of the cubic lattice at the last step of the decimation procedure.

for the Boltzmann weight W but not for this Boltzmann weight multiplied by one of the Ising spins ($W(K, K', L) \rightarrow \sigma_1 W(K, K', L)$). The corresponding green cube cannot be removed using the disorder condition (2) and (3). Therefore it is no longer possible to sum over the spins at the bottom of the red cube to remove in that way the four red cubes connected to that cube (see figure 3).

This situation is repeated at any step and leads to a pyramid of green cubes. This pyramid is not infinite because of the decimation upwards over the red cubes. Let us concentrate first on the two-point correlation functions. One is led to distinguish between two different situations. In the first situation, one of the points is at the top of the pyramid while the other one is inside. On the contrary in the second case, the other point is outside the pyramid. This is the three-dimensional generalisation of the distinction that occurs for the checkerboard models (Dhar and Maillard 1985) where one has to distinguish between the correlation functions that are 'spacelike' and the ones that are not.

Figure 4(a) illustrates the case of a two-point correlation function for which the other point is inside the pyramid (the line joining the two points has even been chosen to be vertical). This two-point correlation function reduces, thanks to this decimation procedure, to a two-point correlation function on a lattice made of an infinite square lattice for which an upward-pointing and a downward-pointing pyramid are glued.

Figure 4(b) illustrates the other situation for which the second point is outside the pyramid having the first one at its top. One can easily be convinced that the decimation not only limits the pyramid but also makes this pyramid not full inside. In that case the two-point correlation is the same as that of an infinite square lattice distorted locally by the surface of the pyramid.

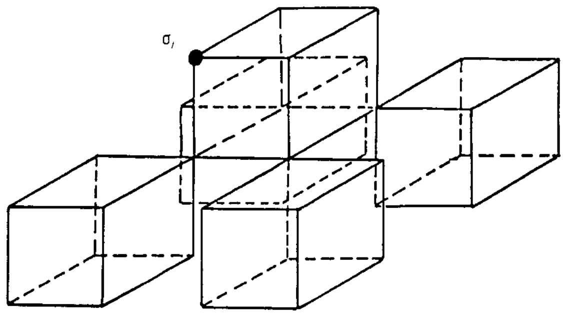


Figure 3. The pyramid of cubes that cannot be removed in the decimation procedure.

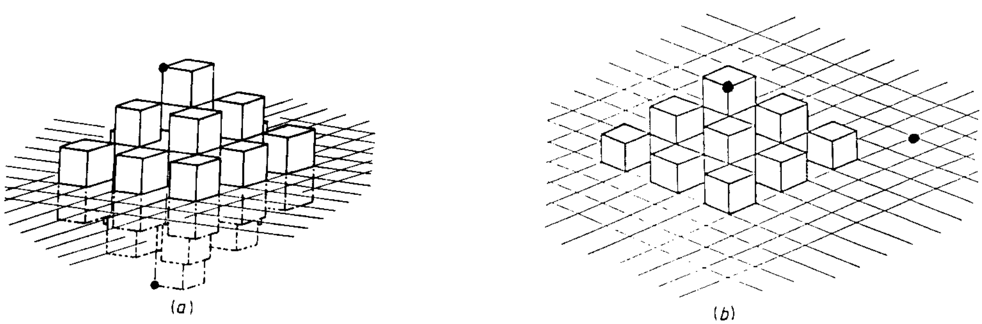
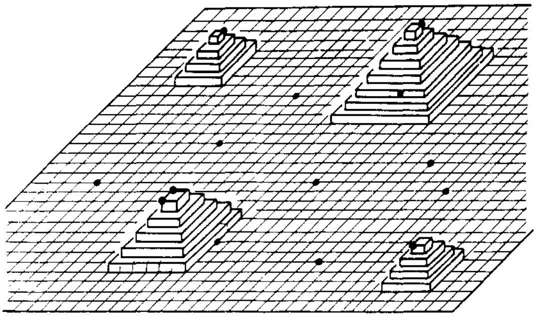
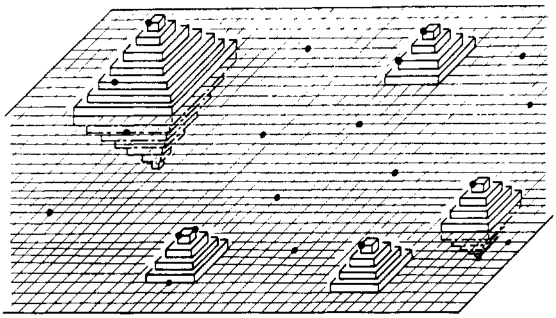


Figure 4. Two-point correlation function and its associated lattice. (a) 'Timelike', (b) 'spacelike'.



(*a*)



(*b*)

Figure 5. N-point correlation function and its associated lattice for cases (a) and (b) described in the text.

Of course, it is a straightforward matter to generalise these results to n-point correlation functions. Figure 5(a) illustrates the case where the n-point correlation functions reduce to n-point correlations on a surface made of the infinite square lattice distorted locally by a finite set of pyramids. The most general situation (figure 5(b)) corresponds to the case where the n-point correlation function reduces to n-point correlation on an infinite square lattice which is deformed locally by a finite set of empty pyramids and on which finite lattices like the one depicted in figure 4 are glued.

3. Intra-plane correlation functions and critical behaviour

When condition (2) is satisfied the intra-plane correlation functions are the same as the correlation functions of a two-dimensional free-fermion model on a square lattice with coupling constants K and K' displayed as indicated in figure 6. Note that this lattice is deduced from a layer of the three-dimensional lattice by changing K' in $-K'$, as a result of the decimation procedure. The calculation of the correlation functions of that particular free-fermion model can, in principle, be performed using Toeplitz determinants (McCoy and Wu 1973) but this is quite tedious: for that reason we deal only with the nearest-neighbour correlation functions. They can be deduced from the partition function of this fully frustrated two-dimensional Ising model by performing a partial derivative with respect to K or K', respectively.

The exact calculation of this partition function has been performed using the well known Vdovitchenko–Feynman counting rules (Vdovitchenko 1965).

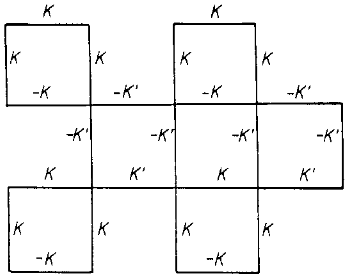


Figure 6. The fully frustrated two-dimensional Ising model with coupling constants K and K'.

Basically it amounts to calculating (for this model) a 16×16 determinant with coefficients depending on K and K'. This determinant, the corresponding result for the partition function, and also the criticality condition, are given in the appendix. When $K = K'$ one verifies immediately that the partition function of the model is the same as the one of the fully frustrated Villain model (Villain 1977).

The critical condition of the model is not an algebraic relation between the two high-temperature variables tanh K and tanh K' but splits into two infinite coupling conditions:

$$(1 - \tanh^2 K)(1 - \tanh^2 K') = 0. \qquad (5)$$

In fact the criticality condition corresponds to K and K' both infinite. If only one of the coupling constants is infinite the model is not critical: it is equivalent to decoupled elementary squares.

We expect both intra-plane ($\xi_{\|}$) and between planes ($\xi_{\perp}$) correlation lengths to diverge at the intersection of (3) with the critical variety and thus (5) must be the exact equation of this intersection. This equation is reminiscent of the criticality condition for the Villain model. As for this model, despite the fact that the critical temperature is zero, the critical exponents are nevertheless non-trivial. One has, for example, $\eta_{\|} = \frac{1}{2}$ (Gabay 1980) and $\nu_{\|} = \infty$. The critical behaviour of the intra-plane correlation functions is thus two dimensional in nature (though with 'exotic exponents') and this is also true of all 'spacelike' correlation functions. On the other hand, we expect correlation functions in the vertical direction to exhibit a genuine three-dimensional critical behaviour with exponents $\nu_{\perp}$ and $\eta_{\perp}$ (corresponding to the transition in the dynamics of the two-dimensional model). At the intersection between the disorder and the critical variety, one thus obtains anisotropic scaling with non-trivial exponents. This occurs despite the fact that the partition function restricted to the disorder variety is perfectly analytic. That such a mechanism is indeed possible without putting constraints on the exponents has been illustrated in Georges *et al* (1986).

4. The general cubic Ising model

It is possible to generalise the previous three-parameter cubic Ising model to a twelve-parameter cubic Ising model with one coupling constant for each bond of the cubic elementary cell (see figure 7(a)). The disorder variety now becomes of codimension 5 (Jaekel and Maillard 1985) and similarly the intra-plane correlation functions reduce, when these five conditions are satisfied, to the correlation functions

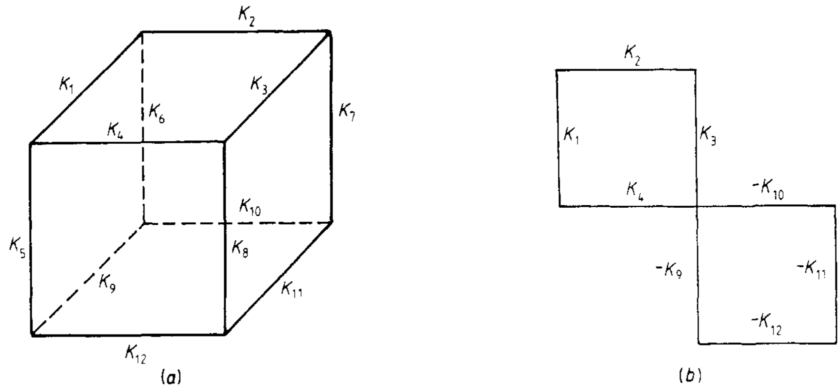


Figure 7. (*a*) The twelve coupling constants of the elementary cubic cell. (*b*) The two-dimensional Ising model with eight coupling constants.

of a two-dimensional free-fermion model with eight coupling constants (see figure 7(*b*)) satisfying an involved condition resulting from the elimination of the vertical coupling constants with the five disorder equations. The partition function of this eight-parameter model and its critical variety have been calculated and it seems that, in this case, one obtains an intersection at non-zero temperature. The exact expression of such a variety is too large to be written in this paper. It is amusing to note that disorder solutions in turn exist for this eight-parameter two-dimensional model. The disorder condition amounts to imposing the two equations:

$$\tanh K_4 + \tanh K_1 \cdot \tanh K_2 \cdot \tanh K_3 = 0 \tag{6a}$$

and

$$\tanh K_{12} + \tanh K_{11} \cdot \tanh K_{10} \cdot \tanh K_9 = 0. \tag{6b}$$

Therefore one can imagine two successive dimensional reductions: one for this model restricted to a certain codimension 5 manifold of the parameter space and a new dimensional reduction restricted to a submanifold of the previous one.

5. The hexagonal close packed Ising model

All these decimation procedures that have been detailed on the cubic lattice can, of course, be applied in a straightforward way to other three-dimensional lattices such as the HCP lattice. A disorder solution exists for the Ising model on this lattice (as found by Welberry and Miller (1978) in a more general case). This disorder solution can also be understood very simply by introducing a local criterion similar to (2) bearing on the Boltzmann weight corresponding to the elementary tetrahedron cell of the lattice (see figure 8). The Boltzmann weight is

$$W(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = \exp(K_1\sigma_i\sigma_j + K_2\sigma_i\sigma_k + K_3\sigma_i\sigma_l + K_{12}\sigma_j\sigma_k + K_{23}\sigma_k\sigma_l + K_{13}\sigma_j\sigma_l. \tag{7}$$

The disorder criterion amounts to writing that, summing over the spin at the top of the tetrahedron, the Boltzmann weight becomes independent of the three other spins:

$$\sum_{\sigma_i} W(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = \lambda(K_1, K_2, K_3, K_{12}, K_{23}, K_{13}). \tag{8}$$

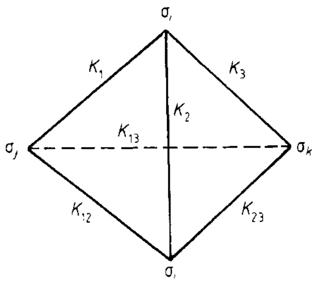


Figure 8. The tetrahedron elementary cell and the four nearest-neighbour coupling constants.

If $K'_{\alpha\beta}$ (K_1, K_2, K_3) denotes the coupling constants obtained from the star-triangle transformation on the coupling constant K_i, the previous disorder condition gives

$$K_{\alpha\beta} + K'_{\alpha\beta}(K_1, K_2, K_3) = 0 \tag{9}$$

thus defining a codimension 3 variety.

The decimation procedure detailed on the cubic lattice just amounts to 'eating' the HCP lattice from the top using the upward pointing tetrahedron and from the bottom using the downward pointing tetrahedron. Under this procedure one will recover the fact that the intra-plane correlation functions are the same as that for the anisotropic triangular Ising model with the coupling constants $K'_{\alpha\beta}$.

These correlation functions are not analytical on the critical variety of this triangular model. Note that, when turning to the variables K_i, the expression of that variety takes the form of the critical variety of an anisotropic hexagonal lattice. This provides the link with the dynamical interpretation of Domany (1984). In slight contrast with the cubic case, the intra-plane correlation functions simply exhibit the standard two-dimensional critical behaviour with $\eta_{\parallel} = \frac{1}{4}$, $\nu_{\parallel} = 1$. One thus has the following situation: in the six-dimensional parameter space of the model there exists a three-dimensional variety for which a dimensional reduction of the model occurs. Some correlation functions of the model are singular for a two-dimensional subvariety of the previous critical variety. The discussion of the phase diagram of such a model has been sketched by many authors (Rujan 1982, Domany and Gubernatis 1985) in the $K_1 = K_2 = K_3$, $K_{12} = K_{23} = K_{13}$ case. Apparently one has an example of a 'Lifshitz surface'.

These drastic simplifications of the n-point correlation functions are a straightforward consequence of the fact that the decimation procedure can be performed from both the top and the bottom of the lattice. This is a remarkable symmetry of the model and it is not satisfied in general, for instance when the Boltzmann weight corresponding to the upward pointing tetrahedron is more complicated than (7). However there exist other interesting classes of disorder solutions: let us impose, for instance, in addition to the disorder condition (8), the following 'linearity' condition on the Boltzmann weight of the upward pointing tetrahedron elementary cell:

$$\sum_{\sigma_i} \sigma_i W(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = \alpha\sigma_j + \beta\sigma_k + \gamma\sigma_l. \tag{10}$$

It can be seen quite easily that two-point (n-point) correlation functions simplify drastically when conditions (8) and (10) are satisfied. Their calculation reduces to a random walk problem in two dimensions.

6. Conclusion

The disorder solutions are known to correspond to particular subvarieties of the parameter space of the model where some dimensional reductions occur. An analysis of three-dimensional Ising models and of their n-point correlation functions has been sketched in this paper. It shows clearly that these dimensional reductions depend on the quantity one deals with: it exhibits a dimensional reduction from dimension three to dimension zero for the partition function to be compared with a dimensional reduction from dimension three to dimension two for an infinite number of correlation functions.

These last results also mean that the study of the model in the vicinity of the disorder variety (2) is much more complicated than in the case of the two-dimensional model (Georges *et al* 1986) where the first-order term of an expansion in the vicinity of (2) was just a simple algebraic expression. For this three-dimensional model even this first-order term is some involved elliptic function (the nearest-neighbour two-point correlation function of a two-dimensional free-fermion model). Finally these precise examples also underline the following points: for the checkerboard Ising model the dimensional reduction associated with the disorder solution reduces the correlation functions of the two-dimensional model to the one of a one-dimensional lattice. Therefore the partition function and the correlation function have no singularities when one restricts the model to the disorder condition for finite values of the parameters (though interesting non-analytic behaviour arise in the region where the disorder and critical varieties are asymptotic one to the other (see Georges *et al* 1986)). This is no longer the case for three-dimensional models where the correlation functions can be non-analytical when one restricts the model to the disorder conditions: the disorder variety can actually contain a non-trivial critical subvariety (a surface in the case of the anisotropic HCP lattice). This generalises the known situation where the disorder variety was just a line and the critical subvariety a Lifshitz tricritical point (Rujan 1982, Domany and Gubernatis 1985). On this critical subvariety, we expect anisotropic scaling to occur with non-trivial critical exponents. Let us also remark that the examples detailed in this paper emphasise the importance of the model for which the decimation procedure can be performed from both the top and the bottom of the lattice. The existence of other interesting classes of disorder solutions ('linear' ones) has been mentioned in the case of the HCP lattice (see also Rujan 1986). Let us finally mention that all these results have a simple dynamical interpretation through the equivalence of 'disorder solutions' of equilibrium spin models with probabilistic cellular automata: this will be the subject of a future paper.

Acknowledgments

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Appendix

The partition function per site of the two-dimensional free-fermion model can be calculated using the Vdovitchenko–Feynman counting rules (Vdovitchenko 1965); this amounts to calculate a 16×16 determinant. For the anisotropic Ising model with two coupling constants corresponding to the horizontal and vertical bonds one deals with a 4×4 determinant. Here the determinant is a 16×16 one because as a consequence of a bigger elementary cell, there are four kinds of different sites on the square lattice denoted by A, B, C, D (see figure 6). The 16×16 matrix is

$$M = \begin{bmatrix} \alpha & 0 & \bar{\omega}\beta & \omega\delta \\ 0 & \gamma & \omega\beta & \bar{\omega}\delta \\ \omega\alpha & \bar{\omega}\gamma & \beta & 0 \\ \bar{\omega}\alpha & \omega\gamma & 0 & \delta \end{bmatrix}$$

where $\omega = \exp(\mathrm{i}\pi/4)$, $\bar{\omega} = \exp(-\mathrm{i}\pi/4)$ and α, β, γ, δ are 4×4 matrices:

$$\alpha = \begin{bmatrix} 0 & -T & 0 & 0 \\ t & 0 & 0 & 0 \\ 0 & 0 & 0 & T \\ 0 & 0 & -t & 0 \end{bmatrix} \mathrm{e}^{\mathrm{i}q_1} \qquad \beta = \begin{bmatrix} 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ T & 0 & 0 & 0 \\ 0 & T & 0 & 0 \end{bmatrix} \mathrm{e}^{\mathrm{i}q_2}$$

$$\gamma = \begin{bmatrix} 0 & t & 0 & 0 \\ -T & 0 & 0 & 0 \\ 0 & 0 & 0 & -t \\ 0 & 0 & T & 0 \end{bmatrix} \bar{\mathrm{e}}^{\mathrm{i}q_1} \qquad \delta = \begin{bmatrix} 0 & 0 & T & 0 \\ 0 & 0 & 0 & T \\ t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \end{bmatrix} \bar{\mathrm{e}}^{\mathrm{i}q_2}$$

where $t = \tanh K$ and $T = \tanh K'$.

The calculation of this determinant has been performed using the formal language REDUCE 3.1 (Hearn 1984). This leads to the following exact expression for the partition function per site

$$\ln Z = 1/2\pi^2 \int\int_0^{2\pi} \mathrm{d}q_1 \, \mathrm{d}q_2 \ln(X(q_1, q_2))$$

where

$$X(q_1, q_2) = A + 4B(\cos q_1 - \cos q_2) + C[2\cos 4q_1 + 2\cos 4q_2 - 4\cos(q_1 + q_2) - 4\cos(q_1 - q_2)]$$

with

$$A = (t^2-1)^2(T^2-1)^2(1 + 2t^2 + 2T^2 + t^4 + T^4 + 8t^2T^2 + 2t^4T^2 + 2t^2T^4 + t^4T^4)$$

$$B = [(t^2-1)^2(t^2+1)t + (T^2-1)^2(T^2+1)T]$$

$$C = (t^2-1)^2(T^2-1)^2 t^2 T^2.$$

The critical variety for this model is obtained when the argument of the logarithm vanishes: this happens when $q_1 = q_2 = 0$. The condition simplifies remarkably and in fact splits into two trivialisation conditions:

$$X(0,0) = 0 \Rightarrow (1 - \tanh^2 K)^4 (1 - \tanh^2 K')^4 = 0.$$

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